Kähler Manifolds and Ricci Curvature

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Contents

1 Preliminaries

A smooth manifold is a topological manifold that is endowed with some sort of **smooth structure**. This means that there exists an atlas of charts that are **smoothly compatible** with each other via transition maps. The structures of these manifolds will vary, but not in ways that are particularly interesting. However, in the holomorphic category, we find that almost everything will change. This section will cover the basics of complex manifolds and their structure.

1.1 Complex Manifolds

Definition 1. *A **complex manifold** is a smooth manifold whose transition maps are **holomorphic**. This means that it is continuous and each of complex-component valued functions have complex partial derivatives with respect to each of the independent complex variables $z_1, ..., z_n$.*

The transition maps being holomorphic here is important as it will introduce a different structure that we will now discuss. If we let M be a $2n-$dimension topological manifold, we let (U, φ) and (V, ψ) denote two coordinate charts defined on M. We say that U and V are **holomorphically compatible** if $U \cap V = \emptyset$ or both transition maps are holomorphic under $\varphi(U \cap V)$, $\psi(U \cap V)$ as open subsets of $\mathbb{C}^n$. A given **holomorphic atlas** is an atlas wherein any two charts within the atlas are holomorphically compatible with each other. A **holomorphic structure** on M is a maximal holomorphic atlas. Then, this finally gives us the complete definition of a complex manifold. An **n-dimensional complex manifold** is a 2n-dimensional topological manifold endowed with a holomorphic structure.

Example 1. *Suppose that M is an n-dimensional complex manifold and let $U \subseteq M$ be an open subset. Then, we can define a canonical holomorphic structure on U given by all of the holomorphic charts of M that contain U. Given this holomorphic structure, U becomes an **open submanifold of M**.*

It is not immediately obvious how to define functions and differentials on these new manifolds that we have defined. To do so, we need to extend some of the ideas of real smooth manifolds to complex ones. This is where we need to be able to "complexify" our spaces somehow.

1.2 Complexification and Almost Complex Structures

1.2.1 Complexification

Given a complex function

$$f = u + iv$$

We want to write its differential as

$$df = du + idv$$

However, this is not our usual one-form(as we have encountered it in real manifold theory). Thus, this forces us to make the following definition

Definition 2. *Suppose V is a real vector space, the* ***complexification of V****, denoted by $V_{\mathbb{C}}$, is the vector space $V \oplus V$ with multiplication given by*

$$(a+ib)(u,v) = (au - bv, av + bv) \qquad a + ib \in \mathbb{C}$$

addition is defined as usual. This turns $V_{\mathbb{C}}$ into a vector space over $\mathbb{C}$.

Note that we will have an isomorphism from V onto the subspace $V \oplus \{0\} \subseteq V_{\mathbb{C}}$, given by the mapping $\psi : u \mapsto (u, 0)$. This means that we can identify V with its image under ψ meaning that we can consider V to be a real-linear subspace of $V_{\mathbb{C}}$. Under this identification, we can then view $V_{\mathbb{C}}$ to be all the linear combinations of V with complex coefficients. From this, we can also determine a basis for $V_{\mathbb{C}}$. If we have a basis for V given by $b_1, ..., b_n$, then a basis for $V_{\mathbb{C}}$ is given by $\{(b_1, 0), ..., (b_n, 0)\}$. For example, this means that $\mathbb{R}^n$ can be identified with $\mathbb{C}^n$. Note that this idea of complexification can extend to linear mappings as well.

Definition 3. *Suppose that $f : V \to W$ is a linear map between real vector spaces. Then, this will extend to the* ***complexification of f****, a complex-linear map $f_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ such that $f_{\mathbb{C}}(u + iv) = f(u) + if(v)$.*

Note that the complexification of a vector space V and a linear map f gives us a functor from the real-valued vector spaces to the complex-valued ones. This can be seen by the following diagram

$$\begin{array}{ccc}
V & \xrightarrow[\text{complexification}]{} & V_{\mathbb{C}} \\
\Big\downarrow f & & \Big\downarrow f_{\mathbb{C}} \\
W & \xrightarrow[\text{complexification}]{} & W_{\mathbb{C}} \\
\Big\downarrow f' & & \Big\downarrow f'_{\mathbb{C}} \\
U & \xrightarrow{\text{complexification}} & U_{\mathbb{C}}
\end{array}$$

Now, we are well-equipped to define a vector bundle in the complex sense.

Definition 4. *Suppose M is a topological space. A* ***complex vector bundle of rank k over M*** *is a topological space E together with a continuous, surjective map $\pi : E \to M$ such that each fiber $E_p = \pi^{-1}(p)$ has the structure of a k-dimensional complex vector space. Furthermore, for each $p \in M$, there exists a neighborhood U over which there exists a local trivialization. This is a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ which restricts to an isomorphism $E_q \to \{q\} \times \mathbb{C}^k$, $q \in U$.*

1.2.2 Almost Complex Structures

From here, we want to be able to begin defining complex structures on tangent spaces, as the end goal is to establish what it means to have a "holomorphic tangent bundle". To do this, we need to have some more tools and definitions at our disposal.

Definition 5. *Suppose that V is a vector space over $\mathbb{R}$. Then a* ***complex structure*** *on V is a real-linear injective homomorphism $J : V \to V$ such that $J \circ J = -Id$.*

Proposition 1. ***Properties of complex structures.*** *Suppose V an n-dimensional $\mathbb{R}$ vector space equipped with a complex structure J and $V_{\mathbb{C}}$ is the complexification. Then*

1. *J is an isomorphism*
2. *The eigenvalues of J are +i, -i*
3. *J is diagonizable*
4. *If V' an eigenspace of +i and V" the eigenspace of -i, then $V_{\mathbb{C}} = V' \oplus V''$.*

Proof.

1. We do this by showing both injectivity and surjectivity. To show injectivity:

$$v \in \ker J \implies -v = J(J(v)) \text{ (by def of J) } = J(0) = 0 \implies v = 0$$

To show surjectivity, suppose that $v \in V_{\mathbb{C}}$. Let $w = -J(v)$:

$$J(w) = J(-J(v)) = -J(J(v)) = v$$

2. Let $\lambda \in \mathbb{C}$ be an eigenvalue of J. Then for $v \in V_{\mathbb{C}}$

$$-v = J(J(v)) = J(\lambda v) = \lambda J(v) = \lambda^2 v$$

which implies that $\lambda^2 = 1$.

3. To do this we just claim that $(x + i)(x - i)$ is the minimal polynomial of J.

$$(J + i)(J - i) = J^2 - ij + ij - i^2 = -\text{id} + \text{id} = 0$$

4. This follows directly by what we have proved in (1)-(3)

□

Definition 6. *Let M be a smooth manifold and suppose that J is a smooth $(1, 1)$ tensor field on M. Let $p \in M$ and suppose $J(p) \in End_{\mathbb{R}}(T_pM)$. For all $p \in M$, $J(p)^2 = -id_{T_pM}$ J is an* ***almost complex structure*** *on M*

This definition is saying that a manifold M with a complex structure J defined on its tangent bundle is an almost complex structure on M. We talk more about tangent bundles below.

1.2.3 Complexified Tangent Bundles and the Holomorphic Tangent Bundle

We can now begin to define the idea of a holomorphic tangent bundle. To do so, we start by defining what a complexified tangent space is. Suppose that M is a complex manifold. Let $p \in M$. Let U be a neighborhood about p and let D be an open subset of $\mathbb{C}^n$. Then, there exists an isomorphism $f : U \to D$ such that $f(p) = 0$. This gives us a local holomorphic coordinate system $z_1, ..., z_n$ centered about p. Suppose that x_j, y_j are smooth, real-valued functions on U. We can then write $z_j = x_j + iy_j$. This then gives us an isomorphism $h : U \to \mathbb{R}^{2n}$. Then in this setting we are ready to define the tangent spaces.

Definition 7. *The* ***real tangent space*** *at the point p is given by*

$$T_pM = \mathbb{R}\left\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n}\right\}$$

Then, the ***complexified tangent space*** *at point p is given by*

$$T_{p,\mathbb{C}}M = \mathbb{C}\left\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n}\right\}$$

Alternatively, the complexified tangent space at point p can be written as

$$T_{p,\mathbb{C}}M = \mathbb{C}\left\{\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, ..., \frac{\partial}{\partial \bar{z}_n}\right\}$$

where

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

Writing the complexified tangent space in this way gives rise to the following definition.

Definition 8. *The two subspaces*

$$T'_pM = \mathbb{C}\left\{\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}\right\} \qquad T'_pM = \mathbb{C}\left\{\frac{\partial}{\partial \bar{z}_1}, ..., \frac{\partial}{\partial \bar{z}_n}\right\}$$

are called the ***holomorphic*** *and* ***antiholomorphic*** *tangent spaces respectively. These two subspaces will gives us the following decomposition*

$$T_{\mathbb{C},p}M = T'_pM \oplus T''_pM$$

Now, we have the tools to begin defining holomorphic tangent bundles.

Definition 9. *The holomorphic tangent spaces T'_pM are the fibers of a holomorphic vector bundle $T'M$. This is the* ***holomorphic tangent bundle*** *of M.*

To see this, we need to be able to describe a set of transition functions for the tangent bundle. Suppose that the dimension of M is n and cover M by coordinate charts $\varphi_\alpha : U_\alpha \to D_\alpha$ where $D_\alpha \subseteq \mathbb{C}^n$ open. Let the following be the transition maps between the charts

$$\psi_{\alpha,\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

Then, its differential is then a holomorphic mapping from $\varphi_\beta(U_\alpha \cap U_\beta)$ into $GL_n(\mathbb{C})$. Denote this differential by $J(h_{\alpha,\beta})$. Then, we claim the following are the transition functions for $T'M$.

$$g_{\alpha,\beta} = J(h_{\alpha,\beta}) \circ \varphi_\beta$$

To verify this we need to check the compatability conditions.

$$\begin{aligned} g_{\alpha,\beta} \cdot g_{\beta,\gamma} &= (J(h_{\alpha,\beta} \circ \varphi_\beta)) \cdot (J(h_{\beta,\gamma}) \circ \varphi_\gamma) \\ &= ((J(h_{\alpha,\beta}) \circ h_{\beta,\gamma})) \\ &= J(h_{\alpha,\beta} \circ h_{\beta,\gamma}) \circ \varphi_\gamma \\ &= J(h_{\alpha,\gamma}) \circ \varphi_\gamma = g_{\alpha,\gamma} \end{aligned}$$

Thus, $g_{\alpha,\beta}$ are indeed the transition functions for a holomorphic vector bundle $\pi : T'M \to M$.

Proposition 2. *Suppose V is a finite-dimensional complex vector space with its standard holomorphic structure. For each $a \in V$, there is a canonical, basis independent, complex linear isomorphism*

$$\Phi_a : V \cong T'_a V$$

In particular, this isomorphism is natural. If $L : V \to W$ is a complex-linear map between finite-dimensional complex vector spaces, then the folowing diagram will commute for each $a \in V$.

$$\begin{array}{ccc} V & \xrightarrow{\Phi_a} & T'_a V \\ \Big\downarrow{\scriptstyle L} & & \Big\downarrow{\scriptstyle D'L(a)} \\ W & \xrightarrow[\Phi_{L(a)}]{} & T'_{L(a)} W \end{array}$$

Proof. Let $a, w \in V$. We define the following holomorphic map.

$$\lambda_{a,w} : \mathbb{C} \to V \qquad \tau \mapsto a + \tau w$$

We define the mapping $\Phi_a : V \to T'_a V$ by the following

$$\Phi_a(w) = D'(\lambda_{a,w}(0)) \left(\frac{\partial}{\partial \tau}|_0 \right) \tag{1}$$

From this definition, we can see that this is independent of any choice of basis for V. Thus, we choose a basis for V and let $(z^1, ..., z^n)$ be the corresponding linear coordinates. Then Φ_a has the coordinate representation

$$\Phi_a(w^1, ..., w^n) = w^j \frac{\partial}{\partial z^j}|_a$$

which gives us a complex-linear isomorphism. Let W be a finite-dimensional complex vector space and $L : V \to W$ be a complex-linear map. Then using any linear coordinates $(\zeta^1, ..., \zeta^m)$ for W we see that

$$\begin{aligned} D'L(a)(\Phi_a(w^1, .., w^n)) &= L^j_k w^k \frac{\partial}{\partial \zeta^j}|_{L(a)} \\ &= \Phi_{L(a)}(L(w^1, ..., w^n)) \end{aligned}$$

which shows that the diagram commutes. □

As a summary, we have defined the following new(and old) bundles arising from complex manifold structures.

1. **TM**: this is the familiar tangent bundle of a smooth manifold, and it is a real vector bundle of rank $2n$.
2. **T$_\mathbb{C}$M**: this is the complexified tangent bundle, which is a complex vector bundle of rank $2n$.
3. **T'M**: this is the holomorphic tangent bundle, which is a complex vector subbundle of $T_\mathbb{C}M$ of rank n. At each point of its fiber, it is the $i-$eigenspace of J_M.
4. **T"M**: this is the antiholomorphic tangent bundle, which is the second complex vector subbundle of $T_\mathbb{C}M$ of rank n. At each point of its fiber it is the $-i$-eigenspace of J_M.
5. **T$_J$M**: this is the ordinary tangent bundle of M with the complex structure J_M, turning it into a complex vector bundle of rank n.

2 Kähler Metrics and Manifolds

Now that we have defined complex manifolds, and some complex structures that arise from them, it is also important to examine the relationship between complex structures and metrics on a tangent bundle. To do this, we introduce Kähler metrics as a special case of Hermitian metrics and then study the manifolds that admit the Kähler metric. To do this, we must first introduce what a Hermitian metric is.

2.1 Hermitian Metrics

Definition 10. *Suppose M is a complex manifold with an almost complex structure J. Then, the **Hermitian fiber metric** on $T_J M$ is a map*

$$h : \Gamma(T_J M) \times \Gamma(T_J M) \to C^\infty(M; \mathbb{C})$$

such that the following properties are satisfied

1. *h is bilinear over $C^\infty(M; \mathbb{R})$*
2. *$h(JX, Y) = ih(X, Y)$*
3. *$h(X, JY) = -ih(X, Y)$*
4. *$h(Y, X) = \overline{h(X, Y)}$*
5. *$h(X, X) > 0$ when $X \neq 0$*

Now that we have the definition of a Hermitian fiber metric, we want to have a deeper understanding of it. We first examine the real part of a Hermitian fiber metric.

Lemma 1. *If M is a complex manifold and h is a Hermitian fiber metric on $T_J M$ then $g = Re\ h$ is a Riemannian metric on M.*

Proof. Note that g is defined to be smooth, positive definite, and bilinear over $C^\infty(M; \mathbb{R})$. Thus, we just need to show that it is symmetric.

$$\begin{aligned} g(X, Y) &= \frac{1}{2}(h(X, Y) + \overline{h(X, Y)}) \\ &= \frac{1}{2}(h(X, Y) + h(Y, X)) \end{aligned}$$

□

Now we move on to examining the imaginary part.

Lemma 2. *Suppose M a complex manifold and h a Hermitian fiber metric on $T_J M$ then ω = -Im h is a $2-$form of type $(1, 1)$.*

Proof. To do this we just need to prove antisymmetry and that ω is indeed of type $(1, 1)$. Antisymmetry follows by the following

$$\begin{aligned} \omega(X, Y) &= \frac{1}{2i}(h(X, Y) - \overline{h(X, Y)}) \\ &= \frac{-1}{2i}(h(X, Y) - h(Y, X)) \end{aligned}$$

which will change in sign if Y and X are swapped. To show that ω is of type $(1,1)$, we have that from how h is defined that $h(JX, JY) = (i)(-i)h(X, Y) = h(X, Y)$ for all real vector fields X, Y. Then

$$\begin{aligned}\omega(JX, JY) &= -\text{Im } h(JX, JY)\\ &= -\text{Im}h(X, Y)\\ &= \omega(X, Y)\end{aligned}$$

□

Now that we have defined a Hermitian fiber metric, we can define the Hermitian metric.

Definition 11. *A* ***Hermitian metric*** *on a complex manifold M is a Riemannian metric where J is orthogonal. A manifold that is equipped with a Hermitian metric is called a* ***Hermitian manifold****. If we have a Hermitian manifold (M, g), the $2-$form given by $\omega = g(J\cdot, \cdot)$ is the* ***fundamental two-form*** *of the Hermitian metric*

Similar to the Riemannian metric, we have the following result for the Hermitian metric.

Lemma 3. *Every complex manifold can be endowed with a Hermitian metric*

Proof. Let M be a complex manifold and suppose that g_0 is an arbitrary Riemannian metric on M. Suppose that g is another Riemannian metric given by

$$g(X, Y) = g_0(X, Y) + g_0(JX, JY)$$

but then note that $g(JX, JY) = g(X, Y)$ and we are done. □

Now that we have defined what a Hermitian metric is, we can define the Kähler metric.

2.2 The Kähler Metric and Kähler Manifold

Definition 12. *A* ***Kähler metric*** *on a complex manifold is a Hermitian metric where its fundamental $2-$form ω is closed. A complex manifold endowed with a Kähler metric is a* ***Kähler manifold****.*

Note that every Hermitian metric can be determined by its fundamental 2-form, meaning that we can also define a Kähler metric using it.

Definition 13. *A* ***Kähler form*** *on a complex manifold is a smooth, closed, positive (1,1)-form.*

Note that every Kähler form will determine a Kähler metric: $g = \omega(\cdot, J\cdot)$. Furthermore, since a Kähler form is closed and real, it will determine a real cohomology class, the **Kähler class**, $[\omega] \in H^2_{dR}(M; \mathbb{R})$. Now, we proceed to define curvature with Kähler metrics.

2.2.1 Curvature of Kähler Metrics

We have studied the Riemann curvature tensor Rm and its identities. On a Kähler manifold, Rm will have additional symmetries.

Theorem 1. *The curvature tensor of a Kähler metric will satisfy the following symmetries for all $W, X, Y, Z \in \Gamma(T'M)$.*

$$Rm(W, X, \cdot, \cdot) = Rm(\cdot, \cdot, W, X) = 0 \tag{2}$$
$$Rm(\bar{W}, \bar{X}, \cdot, \cdot) = Rm(\cdot, \cdot, \bar{W}, \bar{X}) = 0 \tag{3}$$
$$Rm(W, \bar{X}, Y, \bar{Z}) = Rm(Y, \bar{X}, W, \bar{Z}) \tag{4}$$
$$Rm(W, \bar{X}, Y, \bar{Z}) = Rm(W, \bar{Z}, Y, \bar{X}) \tag{5}$$

Proof. Let Z, W be sections of $T'M$ and U, V complex vector fields. Then, since the Levi-Civita connection will map $\Gamma(T', M)$ to itself we have that

$$R(U, V)Z = \nabla_U \nabla_Z - \nabla_V \nabla_U Z - \nabla_{[U,V]} Z$$

Then, this implies that

$$Rm(U, V, Z, W) = g(R(U, V)Z, W) = 0$$

since g will kill off pairs of sections of $T'M$. This shows the result of (2). Conjugation gives us (3). Furthermore, for sections W, X, Y, Z of $T'M$, the algebraic Bianchi identity will give us

$$\begin{aligned} 0 &= Rm(W, \bar{X}, Y, \bar{Z}) + Rm(\bar{X}, Y, W, \bar{Z}) + Rm(Y, W, \bar{X}, \bar{Z}) \\ &= Rm(W, \bar{X}, Y, \bar{Z}) - Rm(Y, \bar{X}, W, \bar{Z}) \end{aligned}$$

this gives us (4). Applying the identity $Rm(W, X, Y, Z) = Rm(Y, Z, W, X)$ similarly will gives us (5). Now that curvature, and Kähler manifolds have been introduced, we can finally begin to talk about Ricii and Scalar Curvatures. □

3 Ricci and Scalar Curvature

Definition 14. *For a Riemannian manifold, the **Ricci curvature** is the covariant 2-tensor field defined by*

$$Rc(X,) = tr(Z \mapsto R(Z, X)Y)$$

It follows the Riemann curvature tensor is symmetric. In terms of a local frame, it is given the components

$$R_{ab} = R^c_{cab}$$

Definition 15. *The **scalar curvature** is the real-valued function, $S = g^{ab} R_{ab}$ on a local frame, such that it is defined by raising an idenx of R_c and taking the trace.*

Then, we can also express the Ricci and scalar curvatures using holomorphic coordinates. The following are the coordinate expressions

$$Rc = 2R_{j\bar{k}}dz^j dz\bar{z}^k \qquad S = 2g^{j\bar{k}}R_{j\bar{k}} \tag{6}$$

where

$$R_{j\bar{k}} = R^{\bar{m}}_{\bar{m}jk} = R^{-l}_{j\bar{k}l} = -\partial_j\partial_{\bar{k}}\log(\det g)$$

The following result follows directly from this

Lemma 4. *On a Kähler manifold, the Ricci tensor is invariant under J. This means that for all complex vector fields X and Y*

$$Rc(JX, JY) = Rc(X, Y)$$

This gives us the following

Proposition 3. ***The Ricci Form*** *Let (M, G) be a Kähler manifold and Rc its Ricci curvature. We define a 2-tensor field ρ in the following way*

$$\rho(X, Y) = Rc(JX, Y) \tag{7}$$

and is a closed $(1, 1)$ *form. It is called the* ***Ricci form of g***

Proof. We want to show that ρ is antisymmetric, closed, and a $(1, 1)$-form. We use the previous lemma to show that ρ is antisymmetric.

$$\begin{aligned}\rho(X, Y) = Rc(JX, Y) &= Rc(J^2X, JY)\\ &= -Rc(X, JY)\\ &= -Rc(JY, X)\\ &= -\rho(Y, X)\end{aligned}$$

Furthermore, we can express ρ using local holomorphic coordinates

$$\rho = iR_{j\bar{k}}dz^j \wedge d\bar{z}^k = -i\partial_j\partial_{\bar{k}}\log(\det g)dz^j \wedge d\bar{z}^k = -i\partial\bar{\partial}\log(\det g)$$

which is a $(1, 1)$-form. Note also since $\partial \circ \bar{\partial} = d \circ \bar{\partial}$, ρ is also locally exact and so it is closed. □

The Ricci form has a relationship with the Chern connection as exhibited by the following theorem.

Theorem 2. *Let M be a Kähler manifold. Then, the Ricci form is equal to* 2π *times the first Chern form of the Chern connection on* $T'M$*.*

Proof. Suppose ∇' is the Chern connection on $T'M$ with respect to the Hermitian fiber metric. Recall that ∇' is equal to the restriction of the Levi-Civita connection ∇. We want to compute its first Chern form. To do so, we work in holomorphic coordinates $(z^1, ..., z^n)$ and that the connection forms θ^k_j are determined by

$$\theta^l_k(X)\partial_l = \nabla'_X\partial_k = \Gamma^l_{jk}X^j\partial_l$$

which gives us that

$$\theta_k^l = \Gamma_{jk}^l dz^j$$

Then, we have that the Chern form is determined as follows

$$c_1(\nabla') = \frac{i}{2\pi} d\theta_l^l = \frac{i}{2\pi} d(\Gamma_{jl}^l dz^j)$$

so that by what we have done above, this shows that

$$c_1(\nabla') = \frac{i}{2\pi} d\partial \log(\det g) = \frac{i}{2\pi} \bar{\partial}\partial \log(\det g) = \frac{1}{2\pi}\rho$$

□

3.1 The Calabi-Yau Theorem

A big theorem arises from the study of the Ricci form. It was conjectured in 1954 by Eugenio Calabi that if we have a compact Kähler manifold, and ρ is a closed $(1,1)-$form representing the cohomology class $2\pi c_1^{\mathbb{R}}(T'M)$, then there is a Kähler metric in the same Kähler class whose Ricci form is equal to ρ. In other words, Calabi was examining whether or not every form representing $c_1(M)$ is the Ricci form of a specific Kähler metric on M coming from one cohomology class. This conjecture was proved by S.T. Yau twenty years later in 1978. The conjecture has become known as the Calabi-Yau Theorem, and it is stated more formally below.

Theorem 3. ***Calabi-Yau Theorem*** *Let (M,g) be a complex Kähler manifold with Kähler form ω. If ρ is any closed $(1,1)-$form representing $2\pi c_1^{\mathbb{R}}(T'M)$, then there exists a unique Kähler metric on M whose Kähler form is cohomologous to ω and has the same Ricci form as ρ.*

This theorem actually requires quite a lot of PDE theory that would be too much to talk about here. Nonetheless, a consequence of this theorem is the existence of Ricci flat Kähler manifolds, called **Calabi-Yau manifolds**. A metric is **Ricci flat** if its Ricci curvature is zero for every Kähler class. Physicists uses Calabi-Yau manifolds to study super-string theory.

4 References

Most of the content for this expository paper was drawn from John M. Lee's *Introduction to Complex Manifolds.* The part on Complexified Tangent Spaces was based off of Christian Schnell's *Complex Geometry* course notes at Stony Brook. The brief section on the Calabi-Yau theorem was drawn from Mathew George's *Yau's Proof of the Calabi Conjecture* as well as Lee's book.